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Invariant algebraic surfaces of the Rabinovich system

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Abstract
In this paper, we characterize all the Darboux polynomials of the Rabinovich system, $\dot{x} = hy - v_1x + yz$, $\dot{y} = hx - v_2y - xz$ and $\dot{z} = -v_3z + xy$. Moreover, we give the necessary and sufficient conditions in order that the Rabinovich system has a rational first integral or an algebraic first integral.

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1. Introduction and statement of main results

We consider the Rabinovich system

$$\begin{aligned}\dot{x} &= hy - v_1x + yz = P(x, y, z)\\ \dot{y} &= hx - v_2y - xz = Q(x, y, z)\\ \dot{z} &= -v_3z + xy = R(x, y, z)\end{aligned}$$

which is a three-wave interaction model, where x, y and z are real variables, v_1, v_2 and v_3 are the damping rates, and h is proportional to the driving amplitude of the feeder wave; see, for example, [1] or [2]. From the point of view of integrability, this system has been studied using different theories and methods. Using the Painlevé method, Bountis *et al* [2] found three integrals of motion. Applying some algebraic methods, Giacomini *et al* [3] obtained four other integrals of motion. Recently, using the method of characteristic curves for solving linear partial differential equations, Zhang [4] characterized all integrals of motion by computing the Darboux polynomials with constant co-factors, in which we use the fact that a polynomial vector field has a Darboux polynomial f with a constant co-factor k if and only if it has an integral of motion fe^{-kt}; see, for example, [5]. In this paper, applying the method given in [4] we obtain all Darboux polynomials of the Rabinovich system and we provide the necessary and sufficient conditions for the Rabinovich system to have a rational or an algebraic first integral. We remark that the search for Darboux polynomials is a very difficult task. Poincaré [6] said that there are no valid methods to compute Darboux polynomials. Indeed, his statement has been verified in the past century. Here, we provide a method to compute all Darboux polynomials of the system mentioned above. The significance of searching for Darboux polynomials is that we can compute Darboux first integrals by using a sufficient number of Darboux polynomials; see, for example, [7] and [8].

Let $f(x, y, z)$ be a real polynomial in variables x, y and z. The algebraic surface $f(x, y, z) = 0$ is an invariant algebraic surface of the Rabinovich system if

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R = kf \tag{1}$$

for some real polynomial $k(x, y, z)$ of, at most, one degree, which is called the co-factor of $f = 0$. If $f(x, y, z) = 0$ is an invariant algebraic surface, then f is also called a Darboux polynomial. From equation (1) it follows that if an orbit of the Rabinovich system has a point on the invariant algebraic surface $f(x, y, z) = 0$, then the whole orbit is contained in the surface.

We say that a real function

$$H : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \qquad (x, y, z, t) \mapsto H(x, y, z, t)$$

is a first integral of the Rabinovich system, if it is constant on all solution curves ($x(t)$, $y(t)$ and $z(t)$) of the Rabinovich system. In particular, if the first integral H is independent of the time t and is a polynomial, then it is called a polynomial first integral. If the first integral H is independent of the time t and is a rational function, then it is called a rational first integral. If the first integral H is of the form $H = f(x, y, z)e^{\sigma t}$, where $f(x, y, z)$ is a polynomial and σ is a non-zero constant, then it is called an integral of motion.

We say that two first integrals $H_1(x, y, z, t)$ and $H_2(x, y, z, t)$ are independent, if their gradients are linear independent vectors for all points $(x, y, z) \in \mathbb{R}^3$ except perhaps for a set of zero Lebesgue measures. If the Rabinovich system has two independent first integrals, then we say that it is completely integrable. We note that in this case the orbits of the Rabinovich system are contained in the curves $\{H_1(x, y, z, t) = h_1\} \bigcap \{H_2(x, y, z, t) = h_2\}$, where h_1 and h_2 vary in $\mathbb{R}$.

An algebraic function $H(x, y, z) = c$ is a solution of the algebraic equation

$$f_0 + f_1 c + f_2 c^2 + \cdots + f_{n-1} c^{n-1} + c^n = 0$$

where $f_i(x, y, z)$ are rational functions, and n is the smallest positive integer for which such a relation holds. Obviously, any rational function is algebraic. The Rabinovich system is said to be algebraically integrable if it has two independent algebraic first integrals.

Our main results are the following.

Theorem 1. *The Rabinovich system has invariant algebraic surfaces or Darboux polynomials if and only if one of the following statements holds.*

(a) $v_1 = v_2 = v_3 = 0$. *Then* $H_1 = x^2 + y^2 - 4hz$ *and* $H_2 = y^2 + z^2 - 2hz$ *are two polynomial first integrals. Consequently, the Rabinovich system is completely integrable. Their invariant algebraic surfaces are contained in the set* $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - 4hz = c_1, y^2 + z^2 - 2hz = c_2, c_1, c_2 \in \mathbb{R}\}$.

(b) $v_1 = v_2 = v_3 \neq 0$ *and* $h = 0$. *Then* $H_1 = (x^2 + y^2)/(x^2 - z^2)$ *and* $H_2 = (y^2 + z^2)/(x^2 - z^2)$ *are two rational first integrals. Consequently, the Rabinovich system is completely integrable. Their invariant algebraic surfaces are contained in the set* $\big\{(x, y, z) \in \mathbb{R}^3 : c_1(x^2 + y^2) - c_2(x^2 - z^2) = 0, c_3(y^2 + z^2) - c_4(x^2 - z^2) = 0, c_1, \ldots, c_4 \in \mathbb{R}, c_1^2 + c_2^2 \neq 0, c_3^2 + c_4^2 \neq 0\big\}$.

(c) $v_1 = v_2 = 0, v_2 \neq v_3$ *and* $h = 0$. *Then* $H = x^2 + y^2$ *is a polynomial first integral, which is a Darboux polynomial with the zero co-factor.*

(d) *$v_1 \neq v_2$ and $v_2 = v_3 = 0$. Then $H = y^2 + z^2 - 2hz$ is a polynomial first integral, which is a Darboux polynomial with the zero co-factor.*

(e) *$v_1 = v_3 = 0$ and $v_2 \neq 0$. Then $H = x^2 - z^2 - 2hz$ is a polynomial first integral, which is a Darboux polynomial with the zero co-factor.*

(f) *$v_1 = v_2 = v_3 \neq 0$, and $h \neq 0$. Then the Darboux polynomial is $f = x^2 - y^2 - 2z^2$ with the co-factor $-2v_1$.*

(g) *$v_1 = v_2 \neq 0$, $v_2 \neq v_3$, and $h = 0$. Then the Darboux polynomial is $f = x^2 + y^2$ with the co-factor $-2v_1$.*

(h) *$v_1 = v_2 \neq 0$, $v_3 = 2v_1$, and $h \neq 0$. Then the Darboux polynomial is $f = x^2 + y^2 - 4hz$ with the co-factor $-2v_1$.*

(i) *$v_1 \neq v_2$, $v_2 = v_3 \neq 0$, and $h = 0$. Then the Darboux polynomial is $f = y^2 + z^2$ with the co-factor $-2v_2$.*

(j) *$v_1 = v_3 \neq 0$, $v_1 \neq v_2$, and $h = 0$. Then the Darboux polynomials are $f = x + z$ with the co-factor $y - v_1$ and $f = x - z$ with the co-factor $-y - v_1$.*

We note that in the above theorem we obtain two new results compared with theorem 1 of [4], namely the rational first integrals of case (b) and the Darboux polynomials with non-constant co-factors of case (j). From theorem 1 we can obtain the following result.

Corollary 2. *For the Rabinovich system the following statements hold.*

(i) *The Rabinovich system has a rational first integral if and only if $v_1 = v_2 = v_3 = 0$, or $v_1 = v_2 = v_3 \neq 0$ and $h = 0$, or $v_1 = v_2 = 0$, $v_2 \neq v_3$ and $h = 0$, or $v_1 \neq v_2$ and $v_2 = v_3 = 0$, or $v_1 = v_3 = 0$ and $v_2 \neq 0$.*

(ii) *The Rabinovich system is algebraically integrable if and only if $v_1 = v_2 = v_3 = 0$, or $v_1 = v_2 = v_3 \neq 0$ and $h = 0$.*

In order to prove this corollary, we need the following results (see, for example, Goriely [9] and Llibre and Zhang [10]).

Proposition 3. *For the Rabinovich system, the following statements hold.*

(i) *If the polynomial functions f and g are relative prime, then f/g is a rational first integral of the Rabinovich system if and only if f and g are both Darboux polynomials with the same co-factor.*

(ii) *The Rabinovich system is algebraically integrable if and only if it has two independent rational first integrals.*

Corollary 2 follows directly from theorem 1 and proposition 3.

2. The proof of theorem 1

We assume that

$$f(x, y, z) = \sum_{i=0}^{n} f_i(x, y, z)$$

is a Darboux polynomial of the Rabinovich system with a non-constant co-factor $k(x, y, z)$, where f_i is a homogeneous polynomial of degree i for $i = 0, 1, \ldots, n$. We remark that in [4] the author obtained all the Darboux polynomials with constant co-factors. So, here we consider only the case when the co-factor is non-constant. Without loss of generality, we can assume that the co-factor is of the form

$$k(x, y, z) = px + ry + qz + c.$$

Substituting f and k into equation (1) and identifying the terms of the same degree, we obtain

$$yz\frac{\partial f_n}{\partial x} - xz\frac{\partial f_n}{\partial y} + xy\frac{\partial f_n}{\partial z} = (px+ry+qz)f_n \qquad (2)$$

$$yz\frac{\partial f_i}{\partial x} - xz\frac{\partial f_i}{\partial y} + xy\frac{\partial f_i}{\partial z} = (px+ry+qz)f_i + (v_1x-hy)\frac{\partial f_{i+1}}{\partial x} + (v_2y-hx)\frac{\partial f_{i+1}}{\partial y} + v_3z\frac{\partial f_{i+1}}{\partial z} + cf_{i+1} \qquad i=n-1,n-2,\ldots,0. \qquad (3)$$

In what follows, in order to prove our theorem we use the method of characteristic curves for solving linear partial differential equations (see, for example, [11] and [4]). The characteristic equation associated with equation (2) is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{y}{x} \qquad \frac{\mathrm{d}z}{\mathrm{d}y} = -\frac{y}{z}.$$

Its general solution is

$$x^2+y^2=c_1 \qquad y^2+z^2=c_2$$

where c_1 and c_2 are arbitrary constants.

We consider the change of variables

$$u=x^2+y^2 \qquad v=y^2+z^2 \qquad w=y. \qquad (4)$$

Correspondingly, the inverse transformation is

$$x=\pm\sqrt{u-w^2} \qquad y=w \qquad z=\pm\sqrt{v-w^2}. \qquad (5)$$

From equation (2) we obtain the ordinary differential equation (for fixed u and v)

$$-\left(\pm\sqrt{u-w^2}\right)\left(\pm\sqrt{v-w^2}\right)\frac{\mathrm{d}\overline{f}_n}{\mathrm{d}w} = \left[p\left(\pm\sqrt{u-w^2}\right)+rw+q\left(\pm\sqrt{v-w^2}\right)\right]\overline{f}_n$$

where $\overline{f}_n(u,v,w) = f_n(x,y,z)$. In the following, unless otherwise specified, we always denote the function $R(x,y,z)$ by $\overline{R}(u,v,w)$, written in the variables u, v and w by using equation (4).

Solving the last equation we find that for $xz>0$

$$\overline{f}_n = \overline{A}(u,v)\left|2\sqrt{(u-w^2)(v-w^2)}+2w^2-(u+v)\right|^{-r/2} \times \exp\left(-p\left(\pm\arcsin\frac{w}{\sqrt{v}}\right)\right)\exp\left(-q\left(\pm\arcsin\frac{w}{\sqrt{u}}\right)\right)$$

for $xz<0$

$$\overline{f}_n = \overline{A}(u,v)\left|2\sqrt{(u-w^2)(v-w^2)}-2w^2+(u+v)\right|^{r/2} \times \exp\left(-p\left(\pm\arcsin\frac{w}{\sqrt{v}}\right)\right)\exp\left(-q\left(\pm\arcsin\frac{w}{\sqrt{u}}\right)\right)$$

where $\overline{A}(u,v)$ is an arbitrary smooth function in u and v. In order for $f_n(x,y,z)=\overline{f}_n(u,v,w)$ to be a homogeneous polynomial of degree n in x, y and z, we must have $p=q=0$, the function A must be a homogeneous polynomial in x^2+y^2 and y^2+z^2, and r must be a convenient integer. More precisely, if r is a positive or negative integer, then $f_n = (x+z)^rA(x^2+y^2,y^2+z^2)$ or $f_n=(x-z)^{-r}A(x^2+y^2,y^2+z^2)$, respectively. Without loss of generality, we can assume that f is a Darboux polynomial of degree $2m+r$ $(2m-r)$ with the co-factor $k = ry + c$ if r is a positive (negative) integer, and that

$$f = \sum_{i=0}^{2m+r} f_i$$

and

$$f = \sum_{i=0}^{2m-r} f_i$$

respectively, where f_i is a homogeneous polynomial of degree i, and

$$f_{2m+r} = (x+z)^r \sum_{i=0}^{m} a_i^{(m)} (x^2+y^2)^{m-i} (y^2+z^2)^i$$

and

$$f_{2m-r} = (x-z)^{-r} \sum_{i=0}^{m} a_i^{(m)} (x^2+y^2)^{m-i} (y^2+z^2)^i$$

respectively.

First we consider the case $r > 0$. Introducing f_{2m+r} into equation (3) with $i = 2m+r-1$ and performing some calculations, we have

$$\begin{aligned}
yz\frac{\partial f_{2m+r-1}}{\partial x} &- xz\frac{\partial f_{2m+r-1}}{\partial y} + xy\frac{\partial f_{2m+r-1}}{\partial z} - ryf_{2m+r-1} \\
&= (x+z)^r \sum_{i=0}^{m} [(2(m-i)+r)v_1 + 2iv_3 + c] a_i^{(m)} (x^2+y^2)^{m-i} (y^2+z^2)^i \\
&\quad + (x+z)^r \sum_{i=0}^{m-1} 2\left[(m-i)(v_2-v_1)a_i^{(m)} + (i+1)(v_2-v_3)a_{i+1}^{(m)}\right] \\
&\quad \times (x^2+y^2)^{m-i-1}(y^2+z^2)^i y^2 \\
&\quad - (x+z)^r \sum_{i=0}^{m} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j} \\
&\quad \times (x^2+y^2)^{m-1-i+j}(y^2+z^2)^{i-j} xy \\
&\quad + (x+z)^{r-1} \sum_{i=0}^{m} r(v_3-v_1) a_i^{(m)} (x^2+y^2)^{m-i}(y^2+z^2)^i z \\
&\quad - (x+z)^{r-1} \sum_{i=0}^{m} hra_i^{(m)} (x^2+y^2)^{m-i}(y^2+z^2)^i y.
\end{aligned}$$

In the following proof, we consider only the case $xz < 0$. For $xz > 0$ the proof is completely similar. For simplicity, we may assume that $x = \sqrt{u - w^2}$ and $z = -\sqrt{v - w^2}$. Using the transformations (4) and (5), the above equation becomes

$$\begin{aligned}
\sqrt{u-w^2}\sqrt{v-w^2}\frac{\mathrm{d}\overline{f}_{2m+r-1}}{\mathrm{d}w} &- rw\overline{f}_{2m+r-1} \\
&= \left(\sqrt{u-w^2} - \sqrt{v-w^2}\right)^r \sum_{i=0}^{m} [(2(m-i)+r)v_1 + 2iv_3 + c] a_i^{(m)} u^{m-i} v^i
\end{aligned}$$

$$
\begin{aligned}
&+\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \sum_{i=0}^{m-1} 2\left[(m-i)(v_2-v_1)a_i^{(m)}+(i+1)(v_2-v_3)a_{i+1}^{(m)}\right]\\
&\times u^{m-i-1}v^i w^2\\
&-\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \sum_{i=0}^{m} a_i^{(m)} \sum_{j=0}^{1}(4h)^{1-j}\binom{m-i}{1-j}(2h)^j\binom{i}{j}\\
&\times u^{m-1-i+j}v^{i-j}\sqrt{u-w^2}\,w\\
&-\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-1} \sum_{i=0}^{m} r(v_3-v_1)a_i^{(m)}u^{m-i}v^i\sqrt{v-w^2}\\
&-\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-1} \sum_{i=0}^{m} hra_i^{(m)}u^{m-i}v^i w. \qquad (6)
\end{aligned}
$$

This is a linear ordinary differential equation in $\overline{f}_{2m+r-1}$. The corresponding homogeneous equation

$$
\sqrt{u-w^2}\sqrt{v-w^2}\,\frac{\mathrm{d}\overline{f}^*_{2m+r-1}}{\mathrm{d}w}-rw\overline{f}^*_{2m+r-1}=0
$$

has a general solution

$$
\overline{f}^*_{2m+r-1}=\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r\overline{A}^*_{2m-1}(u,v)
$$

where $\overline{A}^*_{2m-1}(u,v)$ is an arbitrary smooth function in u and v. In order to use the method of variation of constants, we assume that

$$
\overline{f}_{2m+r-1}=\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r\overline{A}_{2m-1}(u,v,w)
$$

is a solution of equation (6), then $\overline{A}_{2m-1}(u,v,w)$ satisfies

$$
\begin{aligned}
\frac{\mathrm{d}\overline{A}_{2m-1}}{\mathrm{d}w}=&\sum_{i=0}^{m}[(2(m-i)+r)v_1+2iv_3+c]a_i^{(m)}u^{m-i}v^i\frac{1}{\sqrt{u-w^2}\sqrt{v-w^2}}\\
&+\sum_{i=0}^{m-1}2\left[(m-i)(v_2-v_1)a_i^{(m)}+(i+1)(v_2-v_3)a_{i+1}^{(m)}\right]\\
&\times u^{m-i-1}v\frac{w^2}{\sqrt{u-w^2}\sqrt{v-w^2}}\\
&-\sum_{i=0}^{m}a_i^{(m)}\sum_{j=0}^{1}(4h)^{1-j}\binom{m-i}{1-j}(2h)^j\binom{i}{j}\\
&\times u^{m-1-i+j}v^{i-j}\sqrt{u-w^2}\frac{w}{\sqrt{v-w^2}}\\
&+\sum_{i=0}^{m}r(v_3-v_1)a_i^{(m)}u^{m-i}v^i\frac{1}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)\sqrt{u-w^2}}\\
&-\sum_{i=0}^{m}hra_i^{(m)}u^{m-i}v^i\frac{w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)\sqrt{u-w^2}\sqrt{v-w^2}}.
\end{aligned}
$$

Because

$$\int \frac{w^2\,\mathrm{d}w}{\sqrt{u-w^2}\sqrt{v-w^2}} = -\int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}}\,\mathrm{d}w + u\int \frac{w\,\mathrm{d}w}{\sqrt{u-w^2}\sqrt{v-w^2}}$$

thus

$$\int \frac{\mathrm{d}w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)\sqrt{u-w^2}} = \frac{w}{u-v} + \int \frac{\sqrt{v-w^2}}{\sqrt{u-w^2}}\,\mathrm{d}w$$

and

$$\int \frac{\mathrm{d}w}{\sqrt{u-w^2}\sqrt{v-w^2}} \qquad \int \frac{\sqrt{u-w^2}}{\sqrt{v-w^2}}\,\mathrm{d}w$$

are elliptic integrals of the first and second kind, respectively; see, for example, [12]. In order for $A_{2m-1}(x,y,z) = \overline{A}_{2m-1}(u,v,w)$ to be a homogeneous polynomial of degree $2m-1$, we must have

$$\begin{aligned}
&[(2(m-i)+r)v_1 + 2iv_3 + c]a_i^{(m)} = 0 && i = 0,1,\ldots,m\\
&(m-i)(v_2-v_1)a_i^{(m)} + (i+1)(v_2-v_3)a_{i+1}^{(m)} = 0 && i = 0,1,\ldots,m-1\\
&r(v_3-v_1)a_i^{(m)} = 0 && i = 0,1,\ldots,m.
\end{aligned} \tag{7}$$

Consequently,

$$\begin{aligned}
f_{2m+r-1} = &-(x+z)^r \sum_{i=0}^{m} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j}\binom{m-i}{1-j}(2h)^j\binom{i}{j}\\
&\times (x^2+y^2)^{m-1-i+j}(y^2+z^2)^{i-j}z\\
&-(x+z)^{r-1}\sum_{i=0}^{m} hra_i^{(m)}(x^2+y^2)^{m-i}(y^2+z^2)^i.
\end{aligned}$$

From equation (7) and using the condition $r \neq 0$, for otherwise the co-factor is a constant, we have the following two cases:

(I) $v_1 = v_2 = v_3$, and then $c = -(2m+r)v_1$;
(II) $v_1 = v_3 \neq v_2$, and then $c = -(2m+r)v_1$, and $a_i^{(m)} \neq 0$ for $i = 0,1,\ldots,m$.

Case (I): $v_1 = v_2 = v_3$ *and* $c = -(2m+r)v_1$. Introducing f_{2m+r-1} into equation (3) with $i = 2m+r-2$ and performing some calculations, we obtain

$$\begin{aligned}
&yz\frac{\partial f_{2m+r-2}}{\partial x} - xz\frac{\partial f_{2m+r-2}}{\partial y} + xy\frac{\partial f_{2m+r-2}}{\partial z} - ryf_{2m+r-2}\\
&\quad = (x+z)^r\sum_{i=0}^{m-1} 2hv_1\left[2(m-i)a_i^{(m)} + (i+1)a_{i+1}^{(m)}\right](x^2+y^2)^{m-1-i}(y^2+z^2)^i z\\
&\qquad + (x+z)^r \sum_{i=0}^{m} a_i^{(m)} 2\sum_{j=0}^{2}(4h)^{2-j}\binom{m-i}{2-j}(2h)^j\binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-2-i+j}(y^2+z^2)^{i-j}xyz\\
&\qquad - (x+z)^{r-1}\sum_{i=0}^{m-1} 2h^2r\left[2(m-i)a_i^{(m)} + (i+1)a_{i+1}^{(m)}\right]\\
&\qquad \times (x^2+y^2)^{m-1-i}(y^2+z^2)^i(x-z)y
\end{aligned}$$

$$- (x+z)^{r-1}\sum_{i=0}^{m} hrv_1 a_i^{(m)}(x^2+y^2)^{m-i}(y^2+z^2)^i$$
$$- (x+z)^{r-2}\sum_{i=0}^{m} h^2 r(r-1) a_i^{(m)}(x^2+y^2)^{m-i}(y^2+z^2)^i y.$$

Working in a similar way to the proof of f_{2m+r-1}, from this equation we obtain the ordinary differential equation

$$\begin{aligned}
&\sqrt{u-w^2}\sqrt{v-w^2}\frac{\mathrm{d}\overline{f}_{2m+r-2}}{\mathrm{d}w} - rw\overline{f}_{2m+r-2} \\
&\quad = -\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \sum_{i=0}^{m-1} 2hv_1\left[2(m-i)a_i^{(m)} + (i+1)a_{i+1}^{(m)}\right] \\
&\quad \times u^{m-1-i}v^i\sqrt{v-w^2} \\
&\quad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \sum_{i=0}^{m} a_i^{(m)} 2\sum_{j=0}^{2}(4h)^{2-j}\binom{m-i}{2-j}(2h)^j\binom{i}{j} \\
&\quad \times u^{m-2-i+j}v^{i-j}\sqrt{u-w^2}\sqrt{v-w^2}w \\
&\quad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-1} \sum_{i=0}^{m-1} 2h^2 r\left[2(m-i)a_i^{(m)} + (i+1)a_{i+1}^{(m)}\right] u^{m-1-i}v^i \\
&\quad \times \left(\sqrt{u-w^2}+\sqrt{v-w^2}\right)w \\
&\quad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-1} \sum_{i=0}^{m} hrv_1 a_i^{(m)} u^{m-i}v^i \\
&\quad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-2} \sum_{i=0}^{m} h^2 r(r-1) a_i^{(m)} u^{m-i}v^i w.
\end{aligned}$$

The corresponding homogeneous equation has the general solution

$$\overline{f}^*_{2m+r-2} = \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \overline{A}^*_{2m-2}(u,v).$$

Let

$$\overline{f}_{2m+r-2} = \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \overline{A}_{2m-2}(u,v,w)$$

be a solution of the previous linear ordinary differential equation. Then the function $\overline{A}_{2m-2}$ satisfies the following equation:

$$\begin{aligned}
\frac{\mathrm{d}\overline{A}_{2m-2}}{\mathrm{d}w} &= -\sum_{i=0}^{m-1} 2hv_1\left[2(m-i)a_i^{(m)} + (i+1)a_{i+1}^{(m)}\right] u^{m-1-i}v^i\frac{1}{\sqrt{u-w^2}} \\
&\quad - \sum_{i=0}^{m} a_i^{(m)} 2\sum_{j=0}^{2}(4h)^{2-j}\binom{m-i}{2-j}(2h)^j\binom{i}{j} u^{m-2-i+j}v^{i-j}w \\
&\quad - \sum_{i=0}^{m-1} 2h^2 r\left[2(m-i)a_i^{(m)} + (i+1)a_{i+1}^{(m)}\right] u^{m-1-i}v^i \\
&\quad \times \frac{\left(\sqrt{u-w^2}+\sqrt{v-w^2}\right)w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)\sqrt{u-w^2}\sqrt{v-w^2}}
\end{aligned}$$

$$-\sum_{i=0}^{m} hrv_1 a_i^{(m)} u^{m-i} v^i \frac{1}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)\sqrt{u-w^2}\sqrt{v-w^2}}$$
$$-\sum_{i=0}^{m} h^2 r(r-1) a_i^{(m)} u^{m-i} v^i \frac{w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^2\sqrt{u-w^2}\sqrt{v-w^2}}.$$

In order for $A_{2m-2}(x,y,z) = \overline{A}_{2m-2}(u,v,w)$ to be a homogeneous polynomial of degree $2m-2$, we should have

$$\begin{aligned} &hv_1\left[2(m-i)a_i^{(m)}+(i+1)a_{i+1}^{(m)}\right]=0 \qquad && i=0,1,\ldots,m-1 \\ &hv_1 a_i^{(m)}=0 && i=0,1,\ldots,m. \end{aligned} \tag{8}$$

Therefore,

$$\begin{aligned} f_{2m+r-2} = {} & (x+z)^r \sum_{i=0}^{m-1} a_i^{(m-1)} (x^2+y^2)^{m-1-i}(y^2+z^2)^i \\ & + (x+z)^r \sum_{i=0}^{m} a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j}\binom{m-i}{2-j}(2h)^j\binom{i}{j} \\ & \times (x^2+y^2)^{m-2-i+j}(y^2+z^2)^{i-j} z^2 \\ & + (x+z)^{r-1} \sum_{i=0}^{m} \frac{hr}{2} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j}\binom{m-i}{1-j}(2h)^j\binom{i}{j} \\ & \times (x^2+y^2)^{m-1-i+j}(y^2+z^2)^{i-j}(x-z) \\ & + (x+z)^{r-2} \sum_{i=0}^{m} h^2\binom{r}{2} a_i^{(m)} (x^2+y^2)^{m-i}(y^2+z^2)^i. \end{aligned}$$

where $a_i^{(m-1)}$ is a real constant for $i=0,1,\ldots,m-1$. From conditions (8) we distinguish the following two subcases.

Subcase 1: $h=0$. Then we have

$$f_{2m+r-1} \equiv 0 \qquad f_{2m+r-2} = (x+z)^r \sum_{i=0}^{m-1} a_i^{(m-1)} (x^2+y^2)^{m-1-i}(y^2+z^2)^i.$$

Introducing f_{2m+r-2} into equation (3) with $i = 2m+r-3$ and performing some computations, we obtain

$$\begin{aligned} & yz\frac{\partial f_{2m+r-3}}{\partial x} - xz\frac{\partial f_{2m+r-3}}{\partial y} + xy\frac{\partial f_{2m+r-3}}{\partial z} - ryf_{2m+r-3} \\ & \qquad = -2v_1(x+z)^r \sum_{i=0}^{m-1} a_i^{(m-1)}(x^2+y^2)^{m-1-i}(y^2+z^2)^i. \end{aligned}$$

Working in a similar way to the proof of f_{2m+r-1}, we obtain

$$v_1 a_i^{(m-1)} = 0 \qquad \text{for} \quad i=0,1,\ldots,m-1$$

and $f_{2m+r-3} \equiv 0$. By recursive calculations, we find that for $s=2,3,\ldots,m-1$

$$f_{2m+r-2s} = (x+z)^r \sum_{i=0}^{m-s} a_i^{(m-s)}(x^2+y^2)^{m-s-i}(y^2+z^2)^i \qquad f_{2m+r-2s-1} \equiv 0$$

with conditions

$$v_1 a_i^{(m-s)} = 0 \qquad \text{for} \quad s = 2, 3, \ldots, m-1;\ i = 0, 1, \ldots, m-s.$$

If $v_1 = 0$, then $c = v_1 = v_2 = v_3 = 0$. We find that

$$f = (x+z)^r \sum_{s=0}^{m} \sum_{i=0}^{m-s} a_i^{(m-s)} (x^2+y^2)^{m-s-i} (y^2+z^2)^i$$

is a Darboux polynomial of degree $2m+r$, where $\sum_{i=0}^{m} \left(a_i^{(m)}\right)^2 \neq 0$ and $a_i^{(m-s)}$ is an arbitrary constant for $s = 0, 1, \ldots, m$ and $i = 0, 1, \ldots, m-s$.

If $v_1 \neq 0$, then $a_i^{(m-s)} = 0$ for $s = 1, 2, \ldots, m-1$ and $i = 0, 1, \ldots, m-s$. Hence,

$$f = (x+z)^r \sum_{i=0}^{m} a_i^{(m)} (x^2+y^2)^{m-i} (y^2+z^2)^i$$

is a Darboux polynomial with the non-constant co-factor $k = ry - (2m+r)v_1$, where $\sum_{i=0}^{m} \left(a_i^{(m)}\right)^2 \neq 0$.

Subcase 2: $h \neq 0$ and $v_1 = 0$. Then $c = v_1 = v_2 = v_3 = 0$. Substituting f_{2m+r-2} into equation (3) with $i = 2m+r-3$ and performing some calculations which are similar to the proof of f_{2m+r-1}, we can obtain that

$$\begin{aligned}
f_{2m+r-3} = &-(x+z)^r \sum_{i=0}^{m-1} a_i^{(m-1)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-1-i}{1-j} (2h)^j \binom{i}{j} \\
&\times (x^2+y^2)^{m-2-i+j} (y^2+z^2)^{i-j} z \\
&+ (x+z)^{r-1} \sum_{i=0}^{m-1} hra_i^{(m-1)} (x^2+y^2)^{m-1-i} (y^2+z^2)^i \\
&- (x+z)^r \sum_{i=0}^{m} a_i^{(m)} \sum_{j=0}^{3} (4h)^{3-j} \binom{m-i}{3-j} (2h)^j \binom{i}{j} \\
&\times (x^2+y^2)^{m-3-i+j} (y^2+z^2)^{i-j} z^3 \\
&- (x+z)^{r-1} \sum_{i=0}^{m} hra_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} \\
&\times (x^2+y^2)^{m-2-i+j} (y^2+z^2)^{i-j} xz \\
&+ (x+z)^{r-1} \sum_{i=0}^{m} \frac{h^2 r^2}{2} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j} \\
&\times (x^2+y^2)^{m-1-i+j} (y^2+z^2)^{i-j} \\
&- (x+z)^{r-2} \sum_{i=0}^{m} h^2 \binom{r}{2} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j} \\
&\times (x^2+y^2)^{m-1-i+j} (y^2+z^2)^{i-j} z \\
&- (x+z)^{r-3} \sum_{i=0}^{m} h^3 \binom{r}{3} a_i^{(m)} (x^2+y^2)^{m-i} (y^2+z^2)^i.
\end{aligned}$$

Introducing f_{2m+r-3} into equation (3) with $i = 2m + r - 4$ and performing some computations, we have

$$
\begin{aligned}
&yz\frac{\partial f_{2m+r-4}}{\partial x} - xz\frac{\partial f_{2m+r-4}}{\partial y} + xy\frac{\partial f_{2m+r-4}}{\partial z} - ryf_{2m+r-4}\\
&\quad = (x+z)^r \sum_{i=0}^{m-1} 2a_i^{(m-1)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-1-i}{2-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-3-i+j} (y^2+z^2)^{i-j} xyz\\
&\qquad - (x+z)^{r-1} \sum_{i=0}^{m-1} hra_i^{(m-1)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-1-i}{1-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-2-i+j} (y^2+z^2)^{i-j} (x-z)y\\
&\qquad - (x+z)^{r-2} \sum_{i=0}^{m-1} 2h^2 \binom{r}{2} a_i^{(m-1)} (x^2+y^2)^{m-1-i} (y^2+z^2)^i y\\
&\qquad + (x+z)^r \sum_{i=0}^{m} a_i^{(m)} 4 \sum_{j=0}^{4} (4h)^{4-j} \binom{m-i}{4-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-4-i+j} (y^2+z^2)^{i-j} xyz^3\\
&\qquad + (x+z)^{r-1} \sum_{i=0}^{m} hra_i^{(m)} \sum_{j=0}^{3} (4h)^{3-j} \binom{m-i}{3-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-3-i+j} (y^2+z^2)^{i-j} (3x^2+z^2) yz\\
&\qquad - (x+z)^{r-2} \sum_{i=0}^{m} h^2 r^3 a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-2-i+j} (y^2+z^2)^{i-j} xy\\
&\qquad + (x+z)^{r-2} \sum_{i=0}^{m} 2h^2 \binom{r}{2} a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j}\\
&\qquad \times (2h)^j \binom{i}{j} (x^2+y^2)^{m-2-i+j} (y^2+z^2)^{i-j} xyz\\
&\qquad + (x+z)^{r-2} \sum_{i=0}^{m} h^2 ra_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-2-i+j} (y^2+z^2)^{i-j} yz^2\\
&\qquad - (x+z)^{r-2} \sum_{i=0}^{m} 2h^3 \binom{r}{2} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-1-i+j} (y^2+z^2)^{i-j} y\\
&\qquad - (x+z)^{r-3} \sum_{i=0}^{m} 4h^3 \binom{r}{3} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j}\\
&\qquad \times (x^2+y^2)^{m-1-i+j} (y^2+z^2)^{i-j} xy\\
&\qquad - (x+z)^{r-4} \sum_{i=0}^{m} \frac{h^4}{4} \binom{r}{4} a_i^{(m)} (x^2+y^2)^{m-i} (y^2+z^2)^i y.
\end{aligned}
$$

Working in a similar way to the proof of f_{2m+r-1}, we obtain the ordinary differential equation

$$
\begin{aligned}
&\sqrt{u-w^2}\sqrt{v-w^2}\frac{\mathrm{d}\overline{f}_{2m+r-4}}{\mathrm{d}w} - rw\overline{f}_{2m+r-4} \\
&\quad = -\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \sum_{i=0}^{m-1} 2a_i^{(m-1)} \sum_{j=0}^{2} (4h)^{2-j} \\
&\qquad \times \binom{m-1-i}{2-j} (2h)^j \binom{i}{j} u^{m-3-i+j} v^{i-j} \sqrt{u-w^2}\sqrt{v-w^2}w \\
&\qquad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-1} \sum_{i=0}^{m-1} hra_i^{(m-1)} \sum_{j=0}^{1} (4h)^{1-j} \\
&\qquad \times \binom{m-1-i}{1-j} (2h)^j \binom{i}{j} u^{m-2-i+j} v^{i-j} \left(\sqrt{u-w^2}+\sqrt{v-w^2}\right) w \\
&\qquad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-2} \sum_{i=0}^{m-1} 2h^2 \binom{r}{2} a_i^{(m-1)} u^{m-1-i} v^i w \\
&\qquad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r} \sum_{i=0}^{m} a_i^{(m)} 4 \sum_{j=0}^{4} (4h)^{4-j} \\
&\qquad \times \binom{m-i}{4-j} (2h)^j \binom{i}{j} u^{m-4-i+j} v^{i-j} \sqrt{u-w^2} \left(\sqrt{v-w^2}\right)^3 w \\
&\qquad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-1} \sum_{i=0}^{m} hra_i^{(m)} \sum_{j=0}^{3} (4h)^{3-j} \binom{m-i}{3-j} (2h)^j \\
&\qquad \times \binom{i}{j} u^{m-3-i+j} v^{i-j} \left(3\left(\sqrt{u-w^2}\right)^2 + \left(\sqrt{v-w^2}\right)^2\right) w\sqrt{v-w^2} \\
&\qquad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-2} \sum_{i=0}^{m} h^2r^3a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \\
&\qquad \times \binom{m-i}{2-j} (2h)^j \binom{i}{j} u^{m-2-i+j} v^{i-j} \sqrt{u-w^2}w \\
&\qquad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-2} \sum_{i=0}^{m} 2h^2 \binom{r}{2} a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \\
&\qquad \times \binom{m-i}{2-j} (2h)^j \binom{i}{j} u^{m-2-i+j} v^{i-j} \sqrt{u-w^2}\sqrt{v-w^2}w \\
&\qquad + \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-2} \sum_{i=0}^{m} h^2ra_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \\
&\qquad \times \binom{m-i}{2-j} (2h)^j \binom{i}{j} u^{m-2-i+j} v^{i-j} (v-w^2)w \\
&\qquad - \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-2} \sum_{i=0}^{m} 2h^3 \binom{r}{2} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j}
\end{aligned}
$$

$$\times \binom{m-i}{2-j} (2h)^j \binom{i}{j} u^{m-1-i+j} v^{i-j} w$$
$$-\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-3} \sum_{i=0}^{m} 4h^3 \binom{r}{3} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j}$$
$$\times \binom{m-i}{1-j} (2h)^j \binom{i}{j} u^{m-1-i+j} v^{i-j} \sqrt{u-w^2}\, w$$
$$-\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^{r-4} \sum_{i=0}^{m} \frac{h^4}{4} \binom{r}{4} a_i^{(m)} u^{m-i} v^i w.$$

The corresponding homogeneous equation

$$\sqrt{u-w^2}\sqrt{v-w^2}\frac{\mathrm{d}\overline{f}^*_{2m+r-4}}{\mathrm{d}w} - rw\overline{f}^*_{2m+r-4} = 0$$

has a general solution

$$\overline{f}^*_{2m+r-4} = \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \overline{A}^*_{2m-4}(u, v).$$

Let

$$\overline{f}_{2m+r-4} = \left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^r \overline{A}_{2m-4}(u, v, w)$$

be a solution of the previous linear ordinary differential equation, then $\overline{A}_{2m-4}(u, v, w)$ satisfies the following equation:

$$\frac{\mathrm{d}\overline{A}_{2m-4}}{\mathrm{d}w} = -\sum_{i=0}^{m-1} 2a_i^{(m-1)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-1-i}{2-j} (2h)^j \binom{i}{j}$$
$$\times u^{m-3-i+j} v^{i-j} w$$
$$-\sum_{i=0}^{m-1} hra_i^{(m-1)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-1-i}{1-j} (2h)^j \binom{i}{j}$$
$$\times u^{m-2-i+j} v^{i-j} \frac{\left(\sqrt{u-w^2}+\sqrt{v-w^2}\right) w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)\sqrt{u-w^2}\sqrt{v-w^2}}$$
$$-\sum_{i=0}^{m-1} 2h^2 \binom{r}{2} a_i^{(m-1)} u^{m-1-i} v^i \frac{w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^2 \sqrt{u-w^2}\sqrt{v-w^2}}$$
$$-\sum_{i=0}^{m} a_i^{(m)} 4 \sum_{j=0}^{4} (4h)^{4-j} \binom{m-i}{4-j} (2h)^j \binom{i}{j}$$
$$\times u^{m-4-i+j} v^{i-j} (v-w^2) w$$
$$-\sum_{i=0}^{m} hra_i^{(m)} \sum_{j=0}^{3} (4h)^{3-j} \binom{m-i}{3-j} (2h)^j \binom{i}{j}$$
$$\times u^{m-3-i+j} v^{i-j} \frac{\left(\left(\sqrt{v-w^2}\right)^3 + 3\left(\sqrt{u-w^2}\right)^2 \sqrt{v-w^2}\right) w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)\sqrt{u-w^2}\sqrt{v-w^2}}$$

$$
\begin{aligned}
&-\sum_{i=0}^{m} h^2 r^3 a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} \\
&\times u^{m-2-i+j} v^{i-j} \frac{w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^2 \sqrt{v-w^2}} \\
&-\sum_{i=0}^{m} 2h^2 \binom{r}{2} a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} \\
&\times u^{m-2-i+j} v^{i-j} \frac{w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^2} \\
&+\sum_{i=0}^{m} h^2 r a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} \\
&\times u^{m-2-i+j} v^{i-j} \frac{\sqrt{v-w^2}\, w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^2 \sqrt{u-w^2}} \\
&-\sum_{i=0}^{m} 2h^3 \binom{r}{2} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j} u^{m-2-i+j} v^{i-j} w \\
&-\sum_{i=0}^{m} 4h^3 \binom{r}{3} a_i^{(m)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} (2h)^j \binom{i}{j} \\
&\times u^{m-1-i+j} v^{i-j} \frac{w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^3 \sqrt{v-w^2}} \\
&-\sum_{i=0}^{m} \frac{h^4}{4} \binom{r}{4} a_i^{(m)} u^{m-i} v^i \frac{w}{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^4 \sqrt{u-w^2}\sqrt{v-w^2}}.
\end{aligned}
$$

Using the integrating formula

$$
\begin{aligned}
&\int \frac{\sqrt{v-w^2}\, w \,\mathrm{d}w}{\sqrt{u-w^2}\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^2} \\
&\quad = \frac{1}{(u-v)^2}\left[v w^2 - \frac{w^4}{4} + \frac{w^2+3u-v}{4}\sqrt{u-w^2}\sqrt{v-w^2} \right. \\
&\qquad \left. - \frac{(u-v)^2+2u+6v}{8} \ln \frac{\left(\sqrt{u-w^2}-\sqrt{v-w^2}\right)^2}{2} \right]
\end{aligned}
$$

and the fact that $A_{2m-4}(x, y, z) = \overline{A}_{2m-4}(u, v, w)$ is a homogeneous polynomial in x, y and z, we must have

$$
\begin{aligned}
&4(m-i)(m-i-1)a_i^{(m)} + 4(m-1-i)(i+1)a_{i+1}^{(m)} + (i+2)(i+1)a_{i+2}^{(m)} = 0 \\
&\qquad i = 0, 1, \ldots, m-2
\end{aligned} \tag{9}
$$

where we have applied the equality

$$\sum_{i=0}^{m} a_i^{(m)} \sum_{j=0}^{2} (4h)^{2-j} \binom{m-i}{2-j} (2h)^j \binom{i}{j} (x^2+y^2)^{m-2-i+j} (y^2+z^2)^{i-j}$$
$$= \sum_{i=0}^{m-2} 4h^2 \Big[4(m-i)(m-i-1)a_i^{(m)} + 4(m-1-i)(i+1)a_{i+1}^{(m)}$$
$$+ (i+2)(i+1)a_{i+2}^{(m)} \Big] (x^2+y^2)^{m-2-i} (y^2+z^2)^i.$$

By recursive calculations, we obtain

$$f_{2m+r-2s} = \sum_{k=0}^{s} \sum_{i=0}^{m-k} a_i^{(m-k)} (x^2+y^2)^{m-k-i} (y^2+z^2)^i \binom{r}{2s-2k} h^{2s-2k} (x+z)^{r-2s+2k}$$
$$+ \frac{hr}{2} \sum_{k=0}^{s-1} \sum_{i=0}^{m-k} a_i^{(m-k)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} \binom{i}{j} (2h)^j$$
$$\times (x^2+y^2)^{m-k-1-i+j} (y^2+z^2)^{i-j} \binom{r}{2s-2k-2}$$
$$\times h^{2s-2k-2} (x+z)^{r-2s+2k+2}$$
$$- z \sum_{k=0}^{s-1} \sum_{i=0}^{m-k} a_i^{(m-k)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} \binom{i}{j} (2h)^j$$
$$\times (x^2+y^2)^{m-k-1-i+j} (y^2+z^2)^{i-j} \binom{r}{2s-2k-1}$$
$$\times h^{2s-2k-1} (x+z)^{r-2s+2k+1} \qquad s = 2, 3, \ldots, m$$

$$f_{2m+r-2s-1} = \sum_{k=0}^{s} \sum_{i=0}^{m-k} a_i^{(m-k)} (x^2+y^2)^{m-k-i} (y^2+z^2)^i$$
$$\times \binom{r}{2s-2k+1} h^{2s-2k+1} (x+z)^{r-2s+2k-1}$$
$$+ \frac{hr}{2} \sum_{k=0}^{s-1} \sum_{i=0}^{m-k} a_i^{(m-k)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} \binom{i}{j} (2h)^j$$
$$\times (x^2+y^2)^{m-k-1-i+j} (y^2+z^2)^{i-j} \binom{r}{2s-2k-1} h^{2s-2k-1} (x+z)^{r-2s+2k+1}$$
$$- z \sum_{k=0}^{s} \sum_{i=0}^{m-k} a_i^{(m-k)} \sum_{j=0}^{1} (4h)^{1-j} \binom{m-i}{1-j} \binom{i}{j} (2h)^j$$
$$\times (x^2+y^2)^{m-k-1-i+j} (y^2+z^2)^{i-j} \binom{r}{2s-2k}$$
$$\times h^{2s-2k} (x+z)^{r-2s+2k} \qquad s = 2, 3, \ldots, m$$

$$f_j = 0 \qquad j = 0, 1, \ldots, r-1$$

where the coefficients $a_i^{(s)}$ satisfy the following conditions for $s = 2, 3, \ldots, m-1$

$$4(s-i)(s-i-1)a_i^{(s)} + 4(s-1-i)(i+1)a_{i+1}^{(s)} + (i+2)(i+1)a_{i+2}^{(s)} = 0$$
$$i = 0, 1, \ldots, s-2. \tag{10}$$

From equations (9) and (10) we easily find that for $s = 2, 3, \ldots, m$

$$a_i^{(s)} = (-2)^{i-1}\binom{s-1}{i-1}a_1^{(s)} - (-2)^i\binom{s}{i}a_0^{(s)} \qquad i = 1, 2, \ldots, s.$$

Hence, the Rabinovich system has the Darboux polynomial of degree $2m + r$

$$\begin{aligned} f &= \sum_{l=0}^{2m} f_{2m+r-l} \\ &= (x+z+h)^r \sum_{i=0}^{m}\Big[(x^2 + (2i-1)y^2 + (2i-2)z^2 + 2hi(hr - 2z))a_0^{(i)} \\ &\quad + (y^2 + z^2 + h(hr - 2z))a_1^{(i)}\Big](x^2 - y^2 - 2z^2)^{i-1} \end{aligned}$$

where $a_0^{(i)}, a_1^{(i)}$ are arbitrary constants for $i = 0, 1, \ldots, m$ and $a_1^{(0)} = 0$.

Case (II): $v_1 = v_3 \neq v_2$. Then $c = -(2m + r)v_1$, and $(m - i)a_i^{(m)} + (i + 1)a_{i+1}^{(m)} = 0$ for $i = 0, 1, \ldots, m - 1$. These equations are equivalent to

$$a_i^{(m)} = (-1)^i\binom{m}{i}a_0^{(m)} \qquad i = 0, 1, \ldots, m.$$

Therefore, we have

$$f_{2m+r} = (x+z)^r\sum_{i=0}^{m} a_i^{(m)}(x^2 + y^2)^{m-i}(y^2 + z^2)^i = a_0^{(m)}(x^2 - z^2)^m(x+z)^r$$

$$f_{2m+r-1} = -2hma_0^{(m)}(x^2 - z^2)^{m-1}(x+z)^r z + hra_0^{(m)}(x^2 - z^2)^m(x+z)^{r-1}.$$

Introducing f_{2m+r-1} into equation (3) with $i = 2m + r - 1$ and working in a similar way to solve f_{2m+r-1}, we can prove that

$$\begin{aligned} f_{2m+r-2} = (x+z)^r\sum_{i=0}^{m-1} a_i^{(m-1)}(x^2 + y^2)^{m-1-i}(y^2 + z^2)^i + 4h^2\binom{m}{2}a_0^{(m)}(x^2 - z^2)^{m-2}(x+z)^r z^2 \\ + h^2rma_0^{(m)}(x^2 - z^2)^{m-1}(x+z)^{r-1}(x - z) + h^2\binom{r}{2}a_0^{(m)}(x^2 - z^2)^m(x+z)^{r-2} \end{aligned}$$

with the condition $hv_1 = 0$.

Subcase 1: $h = 0$. Then working in a similar way to the proof of subcase 1 of case (I), we can find that, if $v_1 = 0$, the Darboux polynomial of degree $2m + r$ is of the form

$$f = \sum_{i=0}^{m} a_i(x - z)^i(x + z)^{r+i}$$

where $a_m \neq 0$ and a_i are arbitrary constants for $i = 0, 1, \ldots, m - 1$.

If $v_1 \neq 0$, the Darboux polynomial of degree $2m + r$ is

$$f = a(x - z)^m(x + z)^{r+m}$$

with the co-factor $ry - (2m + r)v_1$, where a is a non-zero constant.

Subcase 2: $h \neq 0$. Then $v_1 = v_3 = c = 0$ and $v_2 \neq 0$. Working in a similar way to the proof of subcase 2 of case (I), we can prove easily that the Darboux polynomial for the Rabinovich system is of the form

$$f = (x+z+h)^r\sum_{i=0}^{m} a_i(x^2 - z^2 - 2hz + rh^2)^i$$

with the co-factor $k = ry$, where a_i are arbitrary constants for $i = 0, 1, \ldots, m$ and $\sum_{i=0}^{m} a_i^2 \neq 0$.

Summing up the above results, we have proven that, if f is a Darboux polynomial with a non-constant co-factor, then one of the following five cases holds:

1. $v_1 = v_2 = v_3 = 0$ and $h = 0$
$$f = (x+z)^r \sum_{s=0}^{m} \sum_{i=0}^{m-s} a_i^{(m-s)} (x^2+y^2)^{m-s-i} (y^2+z^2)^i$$
is a Darboux polynomial with the co-factor $k = ry$.
2. $v_1 = v_2 = v_3 = 0$ and $h \neq 0$
$$f = (x+z+h)^r \sum_{i=0}^{m} \Big[(x^2 + (2i-1)y^2 + (2i-2)z^2 + 2hi(hr-2z))a_0^{(i)} + (y^2 + z^2 + h(hr-2z))a_1^{(i)} \Big] (x^2 - y^2 - 2z^2)^{i-1}$$
is a Darboux polynomial with the co-factor $k = ry$.
3. $v_1 = v_2 = v_3 \neq 0$ and $h = 0$
$$f = (x+z)^r \sum_{i=0}^{m} a_i^{(m)} (x^2+y^2)^{m-i} (y^2+z^2)^i$$
is a Darboux polynomial with the co-factor $k = ry - (2m+r)v_1$.
4. $v_1 = v_3 = 0$ and $v_2 \neq 0$
$$f = (x+z+h)^r \sum_{i=0}^{m} a_i (x^2 - z^2 - 2hz + rh^2)^i$$
is a Darboux polynomial with the co-factor $k = ry$.
5. $v_1 = v_3 \neq 0$, $v_1 \neq v_2$ and $h = 0$
$$f = a(x-z)^m (x+z)^{m+r}$$
is a Darboux polynomial with the co-factor $k = ry - (2m+r)v_1$.

Working in a similar way as in the proof of the case that r is a positive integer, for r being a negative integer we can prove that if f is a Darboux polynomial with a non-constant co-factor, then one of the following five cases holds:

1. $v_1 = v_2 = v_3 = 0$ and $h = 0$
$$f = (x-z)^{-r} \sum_{s=0}^{m} \sum_{i=0}^{m-s} a_i^{(m-s)} (x^2+y^2)^{m-s-i} (y^2+z^2)^i$$
is a Darboux polynomial with the co-factor $k = ry$, where $\sum_{i=0}^{m} \left(a_i^{(m)}\right)^2 \neq 0$ and $a_i^{(m-s)}$ are arbitrary constants for $s = 0, 1, \ldots, m$ and $i = 0, 1, \ldots, m-s$.
2. $v_1 = v_2 = v_3 = 0$ and $h \neq 0$
$$f = (x-z-h)^{-r} \sum_{i=0}^{m} \Big[(x^2 + (2i-1)y^2 + (2i-2)z^2 + 2hi(hr-2z))a_0^{(i)} + (y^2 + z^2 + h(hr-2z))a_1^{(i)} \Big] (x^2 - y^2 - 2z^2)^{i-1}$$
is a Darboux polynomial with the co-factor $k = ry$, where $a_0^{(i)}, a_1^{(i)}$ are arbitrary constants for $i = 0, 1, \ldots, m$, $a_1^{(0)} = 0$ and $\left(a_0^{(m)}\right)^2 + \left(a_1^{(m)}\right)^2 \neq 0$.

3. $v_1 = v_2 = v_3 \neq 0$ and $h = 0$

$$f = (x - z)^{-r} \sum_{i=0}^{m} a_i^{(m)} (x^2 + y^2)^{m-i} (y^2 + z^2)^i$$

is a Darboux polynomial with the co-factor $k = ry - (2m - r)v_1$, where $a_i^{(m)}$ are arbitrary constants for $i = 0, 1, \ldots, m$ and $\sum_{i=0}^{m} \left(a_i^{(m)}\right)^2 \neq 0$.

4. $v_1 = v_3 = 0$ and $v_2 \neq 0$

$$f = (x - z - h)^{-r} \sum_{i=0}^{m} a_i (x^2 - z^2 - 2hz + rh^2)^i$$

is a Darboux polynomial with the co-factor $k = ry$, where a_i are arbitrary constants for $i = 0, 1, \ldots, m$ and $\sum_{i=0}^{m} a_i^2 \neq 0$.

5. $v_1 = v_3 \neq 0$, $v_1 \neq v_2$ and $h = 0$

$$f = a(x + z)^m (x - z)^{m-r}$$

is a Darboux polynomial with the co-factor $k = ry - (2m - r)v_1$, where a is a non-zero constant.

Combining the above conclusions and theorem 1 of [4], we have proven the 'only if' part of the theorem. The 'if' part follows from some easy computations. This completes the proof of theorem 1.

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